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ON THE LINEAR INTEGRALS OF THE CHAPLYGIN EQUATIONS*

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A geometrical property is established of the lines of congruence determined by the first integral of the dynamic equations of the nonholonomic Chaplygin system, with the integral linear with respect to the velocities. In the case of two degrees of freedom and active nonzero forms applied to the system, the property yields explicitly the necessary and sufficient conditions of existence of a linear integral. The result is illustrated by an example.

1. Let us consider a natural nonholonomic system, the equations of perfect constraints of which

$$q^{\cdot\alpha} = \sum_1^n {}_i B_i^{\alpha} q^{\cdot i} \quad (\alpha = n+1, n+2, \ldots, m) \tag{1.1}$$

are such, that the coefficients B_i^l are functions of the first n generalized coordinates q^i of the system only. If the coefficients of kinetic energy T of the system are also independent of the coordinates q^α and the generalized active forces $Q_i = Q_i(q^1, \ldots, q^n), Q_\alpha = 0$ $(i = 1, \ldots, n;\ \alpha = n+1, \ldots, m)$, then the system is called a Chaplygin system /1,2/ and its dynamic equations can be written in the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial q^{\cdot i}} - \frac{\partial T^*}{\partial q^i} - Q_i = \sum_1^n {}_{j,l} N_{ijl}(q^1, \ldots, q^n) q^{\cdot j} q^{\cdot l} \quad (i = 1, \ldots, n) \tag{1.2}$$

Here T^* and the right-hand side of the i-th equation of (1.2) are obtained by substitution into T and

$$\sum_1^n {}_l \sum_{n+1}^m {}_\alpha \frac{\partial T}{\partial q^{\cdot \alpha}} \left(\frac{\partial B_i^\alpha}{\partial q^l} - \frac{\partial B_l^\alpha}{\partial q^i} \right) q^{\cdot l}$$

of the expressions (1.1) to replace $q^{\cdot\alpha}$ $(\alpha = n+1, \ldots, m)$. Clearly,

$$\sum_1^n {}_{i,j,l} N_{ijl} v^i v^j v^l \equiv 0 \tag{1.3}$$

for any n-vector (v^i).

Let the subset of the configurational manifold of the system defined locally by the equations $q^{n+1} = \ldots = q^m = 0$, be a submanifold. We shall denote this submanifold by X_n. Introducing the metric

$$ds^2 = 2T^* dt^2 = a_{ij} dq^i dq^j \quad (\| a^{ij} \| = \| a_{ij} \|^{-1}) \tag{1.4}$$

we transform X_n into a Riemannian manifold. The summation sign in (1.4) and subsequent expressions is omitted, but the repeated indices will be understood to denote summation from 1 to n.

We shall assume that the equations (1.2) admit the first (general) integral of the form

$$\xi_i(q^1, \ldots, q^n) q^{\cdot i} = \text{const} \tag{1.5}$$

Differentiating (1.5) with respect to time we arrive, by virtue of the equations (1.2) and by equating the coefficients of the generalized velocities in the resulting expression to zero, at the expressions

$$\xi_{rk} + \xi_{kr} + \xi^i (N_{irk} + N_{ikr}) = 0 \quad (r, k = 1, \ldots, n) \tag{1.6}$$

$$\xi_i Q^i = 0 \tag{1.7}$$

*Prikl.Matem.Mekhan,45,No.3,466-470,1981 in unknown functions $\xi_1, \ldots, \xi_n$ (we denote by ξ_{rk} the covariant derivatives). The vector field (ξ_i) determines in X_n a line congruence, the equation of which can be written as

$$dq^i/ds = \tau^i(q^1, \ldots, q^n) \quad (i = 1, \ldots, n) \tag{1.8}$$

where

$$\xi_i = \rho\tau_i, \quad \rho^2 = a^{ij}\xi_i\xi_j \quad (i = 1, \ldots, n) \tag{1.9}$$

Substituting (1.9) into (1.6) and (1.7), we can write these equations in the following equivalent form:

$$\rho(\tau_{rk} + \tau_{kr}) + \rho_k\tau_r + \rho_r\tau_k + \rho\tau^i(N_{irk} + N_{ikr}) = 0 \quad (r, k = 1, \ldots, n) \tag{1.10}$$

$$\tau_i Q^i = 0 \quad (\rho_k = \partial\rho/\partial q^k) \tag{1.11}$$

Multiplying (1.10) by $\tau^r\tau^k$ and summing over all values of $r, k = 1, \ldots, n$, and taking into account the conditions (1.3) and

$$\tau_{rk}\tau^r\tau^k = \frac{1}{2}\frac{d}{ds}(\tau_r\tau^r) = 0$$

we find, that the relation $\rho = \text{const}$ represents the integral of (1.8). In this case from (1.10) it follows that

$$\mu_k + \tau^i\tau^r(N_{irk} + N_{ikr}) = \partial\varkappa/\partial q^k, \quad \varkappa = -\ln\rho \quad (k = 1, \ldots, n) \tag{1.12}$$

where the quantities $\mu_k = \tau_{kr}\tau^r$ considered along any line of congruence (1.8) represent the components of the first geodesic curvature vector of this line.

In the case of holonomic constraints (1.1) the tensor (N_{ijl}) is equal to zero, (ξ_i) is the Killing vector, the solution of (1.8) yields the trajectories of the one-parameter group of motions in the Riemannian manifold (X_n, ds^2) and the relations (1.12) express the known fact (/3/, p.608) that the congruence of the lines of curvature of the trajectories in question is normal.

2. Let us set $n = 2$. In addition to the congruence (1.8), we shall consider a congruence orthogonal to it and denote its unit vector by (η_i). Since $\tau^i\eta_i = 0$, then $\tau_{ik}\eta^i = -\tau^i\eta_{ik}$ (η_{ik} are covariant derivatives), therefore multiplying (1.10) by $\eta^r\eta^k$ and performing the summation over all $r, k = 1, \ldots, n$ we obtain the following scalar equation:

$$\eta_{rk}\eta^k\tau^r = \tau^i\eta^r\eta^k N_{irk} \tag{2.1}$$

The value of the left-hand side of (2.1) at some point $(q^1, q^2) \in X_2$ is equal to the curvature (with a sign) of the line of the auxilliary congruence passing through this point. Clearly, the relations (1.12) and (2.1) are equivalent to (1.10), and this proves the following theorem.

Theorem. When $n = 2$, equations (1.2) admit the integral (1.5) when and only when the vector with components (1.12) is potential and conditions (1.11), (2.1) hold.

If $a^{ij}Q_iQ_j \neq 0$, then the relation (1.11) determines uniquely both congruences in question and the theorem yields, in a constructive form, the criterion of existence of the linear first integral of the Chaplygin equations. In the case of holonomic constraints (1.1) the conditions of the theorem reduce to the following two conditions: equation (2.1) shows that the lines of force are geodesic, and the relations (1.12) imply that the curvature of each line orthogonal to the force lines remains constant along this line. As we know /4/, both these conditions are necessary and sufficient for the existence of a linear first integral of the Lagrange equations.

Example. We consider a problem of plane, nonholonomic motion /5/. A solid body rests on the Oxy plane supported at three points: two of these points represent freely sliding feet, and the third point is the point C of contact of a cutting wheel fixed to the body. The wheel cannot slide in the direction perpendicular to its plane. We introduce a rectangular $C\xi\zeta$ coordinate system rigidly attached to the body. The $C\xi$ and $C\zeta$ axes are parallel to the Oxy plane and the $C\xi$ axis is directed along the cutting wheel. Let $(x, y, 0)$ be the coordinates of the point C in the fixed Oxy system, $(0, \zeta)$ the coordinates of the center of mass of the body in the $C\xi\zeta$ system, φ the angle between the $C\xi$ and Ox axes, the mass of the body $M = 1$ and J the central moment of inertia relative to the perpendicular to the Oxy plane. We assume that the active external forces can be reduced to a resultant force parallel to $C\zeta$ and a couple,• the moment of which about the Oz axis is $H(\varphi) \neq 0$. The constraint equation has the form

$$x^{\cdot}\sin\varphi - y^{\cdot}\cos\varphi = 0 \tag{2.2}$$

The configurational manifold of the system is a direct product of the plane R^2 and the circumference. The manifold can be covered e.g. by four coordinate neighborhoods

$$U_1 = R^2 \times \left(-\frac{\pi}{3}, \frac{\pi}{3}\right), \quad U_2 = R^2 \times \left(\frac{\pi}{6}, \frac{5\pi}{6}\right), \quad U_3 = R^2 \times \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right), \quad U_4 = R^2 \times \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$$

The points of every set U_h should be considered separately, but the arguments are the same in each case, therefore we shall limit ourselves to the points belonging to the coordinate neighborhood U_1.

We can write (2.2) at U_1 in the form $y^\cdot = x^\cdot \operatorname{tg} \varphi$. The kinetic energy of the body is

$$T = {}^1/_2 \left[(x^\cdot - \varphi^\cdot \zeta \cos \varphi)^2 + (y^\cdot - \varphi^\cdot \zeta \sin \varphi)^2\right] + {}^1/_2 J \varphi^{\cdot 2}$$

We take, for convenience, the abscissa of the center of mass $q^1 = x - \zeta \sin \varphi$ and the angle $q^2 = \varphi$ as the generalized coordinates. We have

$$y^\cdot = q^{\cdot 1} \operatorname{tg} q^2 + q^{\cdot 2} \zeta \sin q^2, \quad Q^1 = 0, \quad Q^2 = H(q^2)$$

and the system is a Chaplygin system. According to (1.4) we have

$$ds^2 = \frac{1}{\cos^2 q^2} (dq^1)^2 + J (dq^2)^2$$

The Kristoffel symbols are

$$\Gamma_{12}{}^1 = \Gamma_{21}{}^1 = \operatorname{tg} q^2, \quad \Gamma_{11}{}^2 = -\frac{\sin q^2}{J \cos^3 q^2}$$

and the remaining ones are zero. The components of the tensor are

$$N_{112} = N_{121} = \frac{\sin q^2}{2 \cos^3 q^2}, \quad N_{211} = -\frac{\sin q^2}{\cos^3 q^2}$$

and the remaining ones are again zero. From the conditions (1.11) we find $\tau_1 = 1/\cos q^2$, $\tau_2 = 0$ and $\eta_1 = 0$, $\eta_2 = \sqrt{J}$. Consequently $\mu_1 = 0$, $\mu_2 = -\operatorname{tg} q^2$ and the vector the components of which are represented by the left-hand sides of the equations (1.12) is equal to zero, i.e. $\rho \equiv \mathrm{const}$. On the other hand, the curvature of the lines of force $\eta_{rk}\eta^k\tau^r = 0$ and this coincides with the expression in the right-hand side of (2.1). It follows therefore that the equations of motion of the body have the integral

$$q^{\cdot 1}/\cos q^2 = \mathrm{const}$$

which means that the modulus of the velocity of the center of mass of the body remains constant.

3. When $n = 2$, the relations (1.11) and (2.1) represent the necessary and sufficient conditions for the equations (1.2) to admit a particular integral of the form

$$\tau_i q^{\cdot i} = 0 \tag{3.1}$$

Indeed, the definition of the particular integral yields

$$(\tau_{rk} + \tau^i N_{irk}) q^{\cdot r} q^{\cdot k} + \tau_i Q^i \equiv M \tau_i q^{\cdot i} \tag{3.2}$$

where the left-hand side contains the total derivative of (3.1) with respect to time obtained by virtue of the equations (1.2), and the multiplier $M(q, q^\cdot)$ appearing in the right-hand side is not known in advance. It is understood that the multiplier M can only have the form $\nu_i q^{\cdot i}$ where ν_i are unknown functions of the variables $q^1, \ldots, q^n$. The identity (3.2) leads to equations (1.11) and

$$\tau_{rk} + \tau_{kr} + \tau^i (N_{irk} + N_{ikr}) = \nu_r \tau_k + \nu_k \tau_r \quad (r, k = 1, 2) \tag{3.3}$$

Let us set $\omega_1 = \nu_r \eta^r$, $\omega_2 = \nu_r \tau^r$ (η^r are the components of the unit vector of the congruence of the lines of force). We have three equations (3.3). Contracting these equations with $\tau^r\tau^k$, $\tau^r\eta^k$, $\eta^r\eta^k$ we obtain, respectively, the following two equations:

$$\omega_2 = 0, \quad \omega_1 = \tau_{kr}\tau^r\eta^k + \tau^i\tau^r\eta^k (N_{irk} + N_{ikr})$$

and the relations (2.1), Q.E.D.

4. We note that in the above discussions nowhere have we used concrete analytic expressions for the tensors (a_{ij}) and (N_{ijk}), but only the sign definiteness of the matrix $\|a_{ij}\|$ and equation (1.3). Therefore all the results obtained remain valid for the holonomic systems acted upon by forces quadratic in velocities and possessing the properties (1.3).

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